1 a If a solution is not readily seen, use trial and error on the variable with the largest coefficient, as you will expect fewer trials until you find a multiple of the other variable.

Try $x = 0 : 3y = 1$ has no integral solutions.

Try $x = 1 : 11 + 3y = 1$ has no integral solutions.

Try $x = 2 : 22 + 3y = 1$ has the solution $y = -7$.

The HCF of 11 and 3 is 1.

The general solution will be $x = 2 + 3t, y = -7 - 11t, t \in \mathbb{Z}$

b An obvious solution is $x = 1, y = 0$.

The HCF of 2 and 7 is 1.

The general solution will be $x = 1 + 7t, y = -2t, t \in \mathbb{Z}$

Alternatively, if you spot the solution $x = 8, y = -2$, then the general solution will be:

$x = 8 + 7t, y = -2 - 2t, t \in \mathbb{Z}$

c This equation is equivalent to $8x + 21y = 33$, and then the HCF of 8 and 21 is 1.

It is also obvious that y must be odd, and x must be a multiple of 3.

$y = 5$ gives the solution $x = -9$.

$x = -9 + 21t, y = 5 - 8t, t \in \mathbb{Z}$

Alternatively, use a CAS calculator's table feature with $y = -\frac{8}{21}x + \frac{33}{21}$ and find an integer solution. One such is $x = 264, y = -99$, so the general solution can also be written as:

$x = 264 + 21t, y = -98 - 8t, t \in \mathbb{Z}$

d Dividing through by 2 shows that this is the same equation as in part **a**, hence the solution will be the same.

$x = 2 + 3t, y = -7 - 11t, t \in \mathbb{Z}$

e Any even value of y will give a solution.

If $y = 2, x = 4$.

The HCF of this solution is 2.

The general solution will be $x = 4 + 7t, y = 2 - 2t, t \in \mathbb{Z}$

f Dividing through by 5 shows that this is the same equation as in part**e**, hence the solution will be the same.

$x = 4 + 7t, y = 2 - 2t, t \in \mathbb{Z}$

2 From the general solution, when $t = 0$,

$x = 4$ and $y = 2$. If $t \geq 1, y \leq 0$

If $t \leq -1, x < 0$ so only $t = 0$ works.

There is one solution: $x = 4, y = 2$.

3 Let h be the highest common factor of a and b.

a, b and c can be written as $a = hp$,

$b = hq, c = hr + k$, where $0 < k < h$.

The equation becomes $hpx + hqy = hr + k$.

For all integer values of x and y, the left side of the equation will be a multiple of h, while the right side will not be.

Therefore the equation can have no integral solutions.

4 a Let s be the number of spiders and b the number of beetles.

Equating the numbers of legs gives $8s + 6b = 54$.

b This equation simplifies to $4s + 3b = 27$.

$$\begin{aligned} 4s &= 27 - 3y \\ &= 3(9 - b) \\ s &= \frac{3(9-b)}{4} \end{aligned}$$

Solutions will only exist when $9 - b$ is a multiple of 4, and $b > 0, 9 - b > 0$.

This occurs when $b = 1, s = 6$ and when $b = 5, s = 3$.

The answer could be written '3 spiders and 5 beetles, or 6 spiders and 1 beetle'.

5 Equating the value of the coins,

$$\begin{aligned} 20x + 50y &= 500 \\ 2x + 5y &= 50 \\ 5y &= 50 - 2x \\ &= 2(25 - x) \\ y &= \frac{2(25-x)}{5} \\ &= 2\left(5 - \frac{x}{5}\right) \end{aligned}$$

This gives the results as in the table below.

50c coins	0	2	4	6	8	10
20c coins	25	20	15	10	5	0

6 All solutions are given by

$x = 100 + 83t, y = 1 - 19t$

$$\begin{aligned} 100 + 83t &> 0 \\ 83t &> -100 \\ t &> -\frac{100}{83} \end{aligned}$$

$$\begin{aligned} 1 - 19t &> 0 \\ -19t &> -1 \\ t &< \frac{1}{19} \end{aligned}$$

Since t is an integer, $-1 \leq t \leq 0$.

The second solution occurs when $t = -1$.

$$\begin{aligned} x &= 100 - 83 \\ &= 17 \\ y &= 1 + 19 \\ &= 20 \end{aligned}$$

For $19x + 98y = 1998$, one obvious solution is $x = 100, y = 1$.

$x = 100 + 98t, y = 1 - 19t$

$$\begin{aligned} 100 + 98t &> 0 \\ 98t &> -100 \\ t &> -\frac{100}{98} \end{aligned}$$

$$\begin{aligned} 1 - 19t &> 0 \\ -19t &> -1 \\ t &< \frac{1}{19} \end{aligned}$$

Since t is an integer, $-1 \leq t \leq 0$.

The second solution occurs when $t = -1$.

$$\begin{aligned} x &= 100 - 98 \\ &= 2 \\ y &= 1 + 19 \\ &= 20 \end{aligned}$$

7 Equating the value of the notes,

$$\begin{aligned} 10x + 50y &= 500 \\ x + 5y &= 50 \\ x &= 50 - 5y \\ &= 5(10 - y) \end{aligned}$$

This gives the results as in the table below.

$50 notes	0	1	2	3	4	5
$10 notes	50	45	40	35	40	25
	6	7	8	9	10	
	20	15	10	5	0	

8 $\text{Total number of pieces of fruit} = 63x + 7.$

$$\begin{aligned} y &= \frac{63x+7}{23} \\ &= \frac{7(9x+1)}{23} \end{aligned}$$

$$y = \frac{63x+7}{23} = \frac{7(9x+1)}{23}$$

$9x + 1$ must be a multiple of 23.

$$\begin{aligned} 9x + 1 &= 23n \\ 9x &= 23n - 1 \\ x &= \frac{23n-1}{9} \end{aligned}$$

If $n = 2, x = 5$ and $y = 14$.

If $n = 9t + 2$,

$$\begin{aligned} x &= \frac{23n-1}{9} \\ &= \frac{23(9t+2)-1}{9} \\ &= 23t + \frac{23 \times 2 - 1}{9} \\ &= 23t + 5 \\ y &= \frac{7(9x+1)}{23} \\ &= \frac{7((23n-1)+1)}{23} \\ &= 7n \\ &= 7(9t+2) \end{aligned}$$

The next solution will be $x = 28, y = 112$.

The general solution is

$$\begin{aligned} x &= 5 + 23t, \\ y &= 14 + 63t; t \geq 0 \text{ and } t \in \mathbb{Z}. \end{aligned}$$

9 Consider the value of the two types of cattle.

$$\begin{aligned} 410x + 530y &= 10\,000 \\ 41x + 53y &= 1000 \end{aligned}$$

Using a CAS calculator, a spreadsheet, or trial and error,

$x = 5, y = 15$.

5 of the \$410 cattle and 15 of the \$530 cattle.

10 Let the required number be x.

If it leaves a remainder of 6 when divided by 7, then $x = 7n + 6$.

If it leaves a remainder of 9 when divided by 11, then $x = 11m + 9$.

$$7n + 6 = 11m + 9$$
$$7n - 11m = 3$$

One solution is $n = 2, m = 1$.

The general solution is $n = 2 + (-11)t, m = 1 - 7t$.

Replacing t with $-t$ gives $n = 2 + 11t, m = 1 + 7t$.

$t = 0$ gives $n = 2, m = 1, x = 7 \times 2 + 6 = 20$.

The smallest positive number is 20.

The general form is

$$\begin{aligned} x &= 7n + 6 \\ &= 7(2 + 11t) + 6 \\ &= 77t + 20 \text{ for } t \in \mathbb{N} \cup \{0\} \end{aligned}$$

11 Let x be the number of 5-litre jugs used and y the number of 3-litre jugs used.

$$\begin{aligned} 5x + 3y &= 7 \\ 5x &= 7 - 3y \\ x &= \frac{7 - 3y}{5} \end{aligned}$$

Solutions will only exist when $7 - 3y$ is a multiple of 3.

This occurs when $y = -1$:

$$x = \frac{7 + 3}{5} = 2$$

To measure exactly 7 litres, you would pour two full 5-litre jugs into a container and then remove one 3-litre jugful.

12 Obviously the post office can't sell 1c or 2c worth of postage. Nor can it sell 4c or 7c worth, because there's no way to arrange 3c and 5c to get those values.

It can sell 6c worth $(3 + 3 = 6)$ and 8c worth $(3 + 5 = 8)$.

So the problem can be rephrased as $3x + 5y = n, n \geq 8$ where x is the number of 3c stamps and y the number of 5c stamps.

If $n = 8, 3x + 5y = 8$; the obvious solution is $x = 1, y = 1$.

If $n = 9, 3x + 5y = 9$; the obvious solution is $x = 3, y = 0$.

If $n = 10, 3x + 5y = 10$; the obvious solution is $x = 0, y = 2$.

Since this set of three can be made using $3x + 5y$, the next set of three amounts $(11, 12, 13)$ can be made as $3x + 5y + 3$, or by adding another 3c stamp.

Similarly, every set of three consecutive amounts can be made by adding an additional 3c stamp.

Therefore it's possible to create all amounts in excess of 3c, except for 4c and 7c.

13 Consider total cost.

$$1.7a + b = 29.6$$
$$17a + 10b = 296$$

Using a CAS calculator, a spreadsheet, or trial and error,

$a = 8, b = 16$.

8 of type A and 16 of type B.

14 $6x - 9y = 10$ has no integer solutions since the left-hand-side is divisible by 3 while the right side is not.

15 $13k - 18m = 5$

$k = -1$ and $m = -1$ is a solution.

Next solution is $k = -1 + 18$ and $m = -1 + 13$ Therefore the multiple of 13 is 221

16 $10a + b - 5(a + b) = 17$

$5a - 4b = 17$

$a = 5$ and $b = 2$ is a solution.

Therefore one such number is 52

The general solution is $a = 5 + 4n$ and $b = 2 + 5n, n \in \mathbb{Z}$.

A second solution is given by $n = 1$.

That is $a = 9$ and $b = 7$. Therefore one such number is 97